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Modeling and Analysis of Precise Orientation Determination using Theodolite Intersection Method

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Abstract

The theodolite intersection method is a traditional approach in the field of geodetic survey for coordinate measurements. The approach can also be employed as a non-contact technique for industrial dimensional and orientation measurements. In this work, the utilization of double theodolite intersection technique for precise orientation determination was investigated. The technique was mathematically modelled in order to account for errors caused by realistic imperfections of the instruments. The model was developed based on geometric interpretation of covariance-matrix and was simplified by separating the horizontal and vertical computations in order to predict the standard deviations of orientation determinations. The performance of the developed model was evaluated using linearization and Monte Carlo simulation techniques. The evaluation results showed good predictions of standard deviation in the horizontal orientation (azimuth); however, the standard deviation predictions in vertical orientation (elevation) cases were not good due to the separation in horizontal and vertical computations.

Keywords: Theodolite Intersection Method; Double Theodolite Technique; measurement error model.

1 Introduction

Theodolite is a classical measuring instrument used in geodetic survey and construction fields. It is mainly used to measure both vertical and horizontal angles relative to a reference. Electronic-transit theodolites with automatic leveling correction capability are commonly used for precise measurement in the modern days.

The two theodolite intersection method or double theodolite technique is a traditional approach in the field of geodetic survey to the coordinate measurement. The method relies on the triangulation technique to determine three-dimensional coordinate of a geodetic point. The approach is also employed as a non-contact technique for industrial dimensional and orientation measurement [1, 2]. In these modern days, the concept of theodolite intersection is utilized in the applications of computer stereo vision for detecting point in three-dimensional space, autonomous mobile robots, and precise inspections [3–6].

The current work focuses on the application of precise inspection of orientation in which information on both precision and accuracy of measurements are crucial. For the use of double theodolite intersection method for the inspection of object orientation, the precision and accuracy of measurements are depended on the performance of the instruments and the measurement setup. The objectives of this work are primarily to be able to predict the performance of orientation determination based on equipment with known specifications and to be able to determine of equipment specifications that achieve desired measuring criteria.

This paper presents the details of applied mathematics and statistics deployment for a realworld engineering problem as an example for engineering education purposes. The developed, mathematical model of orientation inspections using double theodolite intersection which accounts for imperfection measurement of method is detailed and evaluated. The following sections include the details of problem statement and mathematical modelling. Finally, the results are discussed and conclusions are presented.

2 Problem Statement

The aim of orientation inspection is to precisely determine the azimuth angle, ψ, and elevation angle, θ, of a rigid body. In the inspection set-up, two markers are set on the object body to create reference line for the object orientation. Two theodolites are set to collimate each other and then are used to aim at the markers. Four horizontal angles and four vertical angles are obtained from the process. [7]

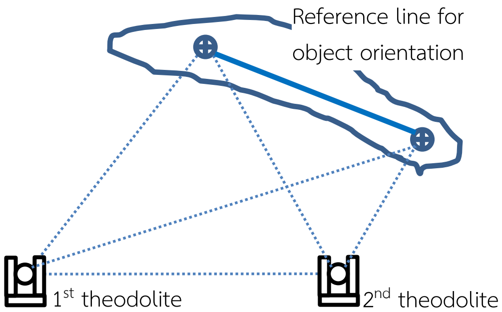


Fig. 1 two theodolite intersection method for object orientation determination

The definition of the orientation parameters is illustrated in Fig. 2. Here, the attitude of rigid body object is defined by a vector from point Q to point P. The definition in Fig. 2 can also be used to define the vertical and horizontal angles measured from a theodolite located at point Q looking at point P.

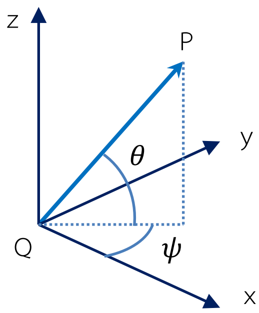


Fig. 2 Orientation definition

Fig. 3 presents the implementation of theodolite intersection technique to determine orientation of an object defined by the vector from point Q to point P. In the figure, two theodolites located at points A and B are used to measure vertical and horizontal angles of the object's reference points (i.e. points P and Q). In Horizontal plane, the four measurements of horizontal angles can be used to determine the azimuth angle of the vector using triangulation technique. After azimuth angle is defined, the elevation angle of the vector is calculated from vertical measurements.

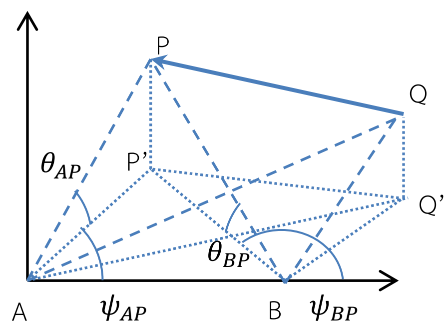


Fig. 3 Implementation of theodolite intersection technique

Consider the theodolites and object reference points in the Cartesian coordinate in which its x- and y-axes are on horizontal plane, a point is defined by

$$\mathbf{x} = [x, y, z]^T \tag{1}$$

For simplification, one theodolite (point A) is set at the origin of local coordinate and another theodolite (point B) is set at a point along x-axis. The measurements of point P are

$$[\psi_{AP}, \theta_{AP}, \psi_{BP}, \theta_{BP}] \tag{2}$$

and the measurements of point Q are

$$[\psi_{AQ}, \theta_{AQ}, \psi_{BQ}, \theta_{BQ}]. \tag{3}$$

The position of point P can be determined by

$$\mathbf{x}_P = \frac{L\sin(\psi_{BP})}{\sin(\psi_{BP}-\psi_{AP})}\begin{bmatrix}\cos\psi_{AP}\\ \sin\psi_{AP}\\ \tan\theta_{AP}\end{bmatrix} \tag{4}$$

where L is the length between points A and B. The position of point Q, $\mathbf{x}_Q$, can also be determined in the same manner.

The azimuth and elevation angles of the vector from point Q to point P can be calculated by

$$\psi_{QP} = \arctan\frac{y_P - y_Q}{x_P - x_Q} \tag{5}$$

and

$$\theta_{QP} = \arctan\frac{z_P - z_Q}{\|x_P - x_Q, y_P - y_Q\|} \tag{6}$$

As it is well known that all measurements come with errors, the measurement of parameters in Eqs. 2–3 are incorporated with errors. These measurement errors can be represented using statistical representation in the term of accuracy (bias) and precision (standard deviation). Assuming that probability distribution of measurement error is normal, the measurement value are:

$$\theta = N(\bar{\theta}, \sigma_\theta)\,,\quad \psi = N(\bar{\psi}, \sigma_\psi) \tag{7}$$

where $\bar{\theta}, \bar{\psi}$ are the mean of measurement values,

$\sigma_\theta, \sigma_\psi$ are the standard deviation of measurement values.

The problem here is to find a model that receives the set-up and instrument parameters as input and return the uncertainty of determination of azimuth and elevation as outputs. The set-up parameters include the position of object relative to the two theodolites. The instrument parameters are standard deviations in measurement values from the instruments. Here, the error in the set-up (levelling) of theodolite is not considered as modern theodolite mostly equipped with levelling sensors that minimize the error.

3 Methodology

To obtain the uncertainty (standard deviation) of orientation determination using Eqs. 4–6, one may easily deploy Monte Carlo simulations which requires long computation time or linearization techniques which is not very accurate. To obtain the standard deviation of orientation determination fast and accurate, the mathematical model of Eqs. 4–6 is analyzed using applied statistics. The measurements in Eqs. 2–3 are computed from the measurement setup (Fig. 3) and instrument parameters. The outputs are azimuth and elevation of object and the standard deviation of each value. Here, it is considered that the probability distributions of measurement errors from theodolites are normal and are uncorrelated in vertical and horizontal measurements.

3.1 Geometric-covariance-matrix Estimation

In the orientation determination using the theodolite intersection technique, the uncertainties in angle measurements from theodolites in the Polar coordinate system are transformed into the uncertainty in position in the Cartesian coordinate system. Then a reference vector for the object orientation is created using the two points. In this step the uncertainties in position of the two points are combined and finally transformed back to be uncertainties in orientation (angles) in the Polar coordinate system.

Since the vertical and horizontal measurements are uncorrelated, one-standard deviation (1-SD or one-SD) uncertainties of angle measurements from each theodolite would create a point with 1-SD position uncertainty in the shape of Hexahedron. Combining with the position uncertainty of the other point, the 1-SD uncertainty of the vector becomes asymmetric egg shape in three-dimension which is very hard to be determined to is very hard to transform back to 1-SD uncertainties of orientation angles (azimuth and elevation) in the Polar coordinate system.

The Geometric-covariance-matrix Estimation is developed by using covariance matrices to represent uncertainties in the Cartesian coordinate system for operation ease. Each covariance matrix is always symmetry and contains the variances on its diagonal and the covariance off-diagonal. For simplification, the computations of horizontal angles and vertical angles are performed separately. The computation of vertical orientation (elevation) is performed after the computation of horizontal orientation (azimuth) is finished. Fig. 4 presents the concept of 1-SD uncertainty area of joint probability distribution from random variables in x- and y-axes (red dotted line) and the perimeter created from 1-SD range of each variable (solid green line). In the figure, the 1-SD uncertainty area of joint probability distribution can be represented by a 2x2 covariance matrix. So the idea of Geometric-covariance-matrix Estimation is to find a circle or an ellipse that fit (and consume maximum space) inside the perimeter created from 1-SD uncertainties of two random variables.

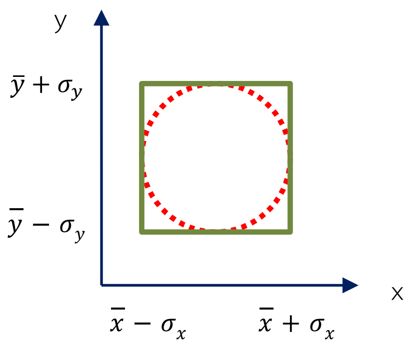


Fig. 4 one-SD perimeter and geometry of corresponding covariance matrix

The perimeter created from 1-SD uncertainties of horizontal angle measurements of each theodolite (1-SD perimeter) at point P (and Q) is normally in the shapes of quadrangles. So the 1-SD uncertainty area of joint probability distribution is estimated using an ellipse as shown in Fig. 5.

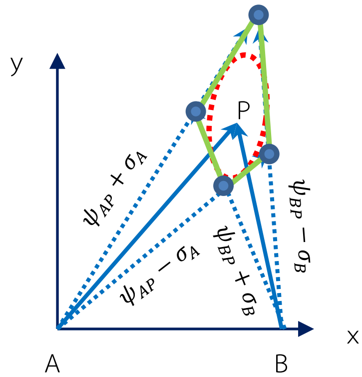


Fig. 5 Position uncertainty from angle measurements in horizontal plane

The ellipse is determined based on four vertices (blue circular marks in Fig. 5) created from the 1-SD perimeter of angle measurements. The ellipse representing uncertainty in the point position is assumed to be the ellipse with largest area inside the 1-SD perimeter of measurements from two theodolites and can be defined by [8].

A covariance matrix associated the general form of uncertainty ellipse can be defined from the major and semi-minor axes as

$$\Sigma_l = \begin{bmatrix} \sigma_a^2 & 0 \\ 0 & \sigma_b^2 \end{bmatrix} \tag{8}$$

and the matrix can be rotated to the reference Cartesian coordinate system as

$$\Sigma_{\text{point}} = R\Sigma_l R^T \tag{9}$$

where $\boldsymbol{R}$ is a rotation matrix.

After points P and Q and their associated uncertainties are obtained, the vector QP is determined by subtraction of points P and Q. With this operation, the uncertainty of vector QP (direction and range) is in the form of the uncertainty in point P position when point Q is fixed at the origin. The uncertainty in vector QP is computed by combining the covariance matrices of the two points assuming there is no cross-correlation as

$$\Sigma_{\text{QP,ij}} = \Sigma_{\text{P,ij}} + \Sigma_{\text{Q,ij}}. \tag{10}$$

The covariance matrix Σ_{QP} is then projected onto the line that is perpendicular to the vector QP in order to transform uncertainty in vector QP direction into the uncertainty in azimuth angle in the Polar coordinate system. Note that the uncertainty in range of vector QP is neglected. Fig. 6 illustrates the described transformation of uncertainty.

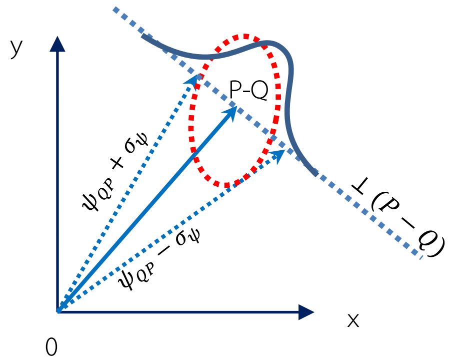


Fig. 6 Transformation of uncertainty in horizontal component of vector QP into uncertainty in azimuth angle

For uncertainty in elevation angles, the 1-SD uncertainty of vertical measurement of the reference theodolite (at the origin point) is transformed into 1-SD uncertainty of vertical position of point P (and Q). The vertical component of vector QP can be computed in the same manner as the horizontal component. The variance of vertical component of vector QP can also be obtained from Eq. 10. Finally the standard deviation of elevation orientation can be computed as shown in Fig. 7.

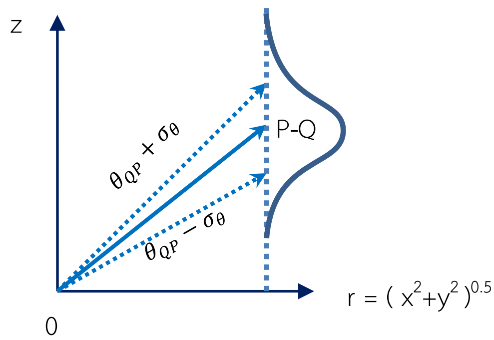


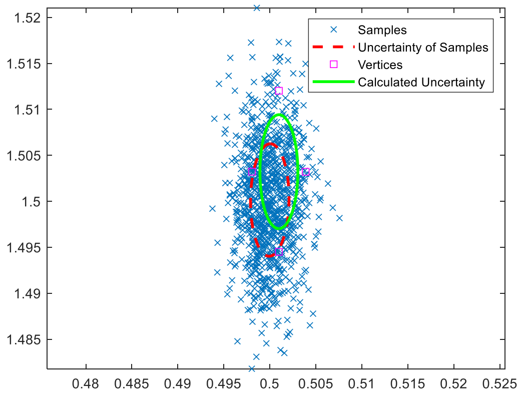
Fig. 7 Transformation of uncertainty in vertical component of vector QP into uncertainty in elevation angle

3.2 Monte Carlo Simulations

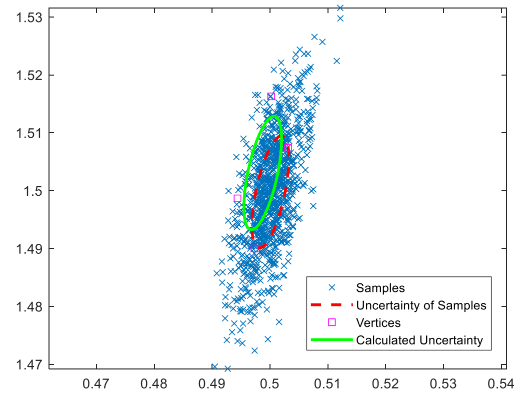
A Monte Carlo simulation is well-known method used to show the probability of different outcomes from a system when includes the intervention of uncertainty. In this paper, the Monte Carlo simulation method is used to present the distribution of azimuth and elevation angle outputs from inputs that incorporate normal distribution errors. The standard deviation of output distribution is computed and used as a reference (a correct value) to evaluate the prediction from developed model.

4 Results and Discussion

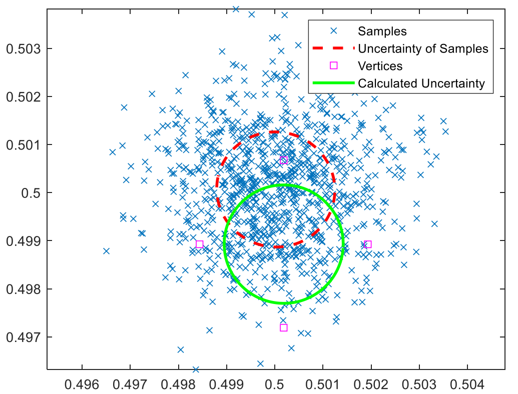
In the investigation of the performance of developed model, the set-up and instrument parameters are specified and used to compute the measurement value with random error. The first theodolite is located at the origin point and the second one is located on x-axis. All of the position values are normalized by the distance between two theodolites. Fig. 8–9 present the comparison between the result from the developed model and that from Monte Carlo simulations when the point is at different distance. For Monte Carlo simulations, 1000 measurements for the same point from both theodolites are randomly generated based on Eq. 7 and used to determine 1000 samples of points (blue markers). The distribution of samples is used to define the uncertainty ellipse for samples (red-dashed line). An uncertainty ellipse of a considered sample obtained from the developed model is presented in the figures (green line). In Fig 8, the standard deviation of both theodolite measurements is 0.1 degree. In Fig 9, the standard deviation of 1st and 2nd theodolite measurements is 0.1 and 0.2 degree, respectively. Fig. 10–12 present the result in which the actual points P and Q are at [0.5, 1.5, 0.7] and [0.9, 1.2, 0.5].



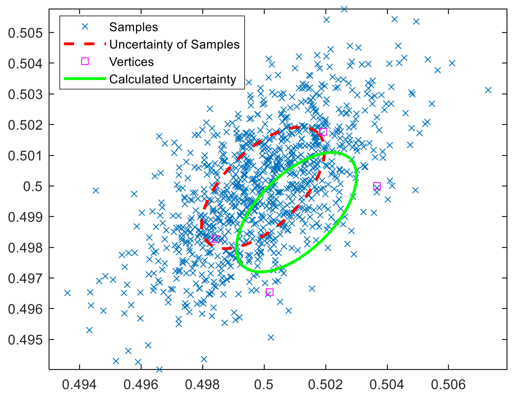
(a) point at 1.5L



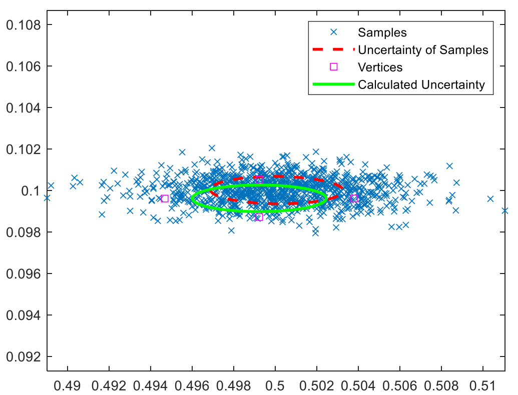
(a) point at 1.5L



(b) point at 0.5L

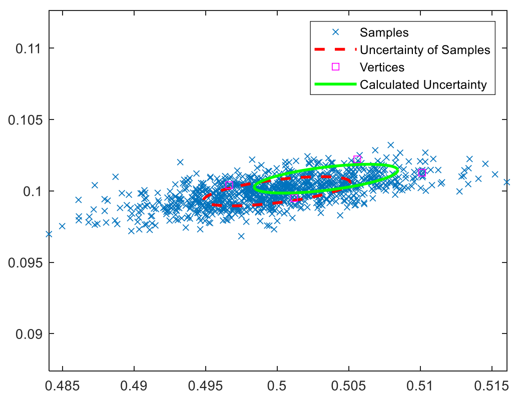


(b) point at 0.5L



(c) point at 0.1L

Fig. 8 uncertainty ellipse for point determination



(c) point at 0.1L

Fig. 9 uncertainty ellipse for point determination

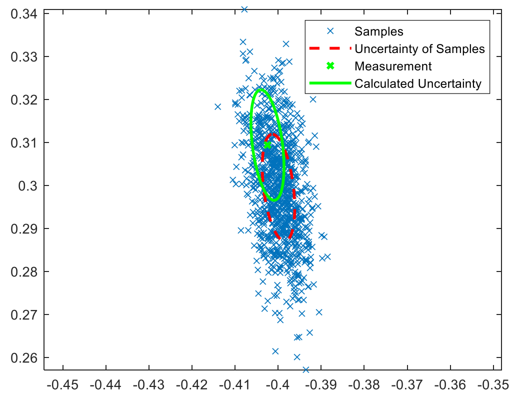


Fig. 10 uncertainty ellipse of vector QP computed from a sample and that computed from 1000 samples

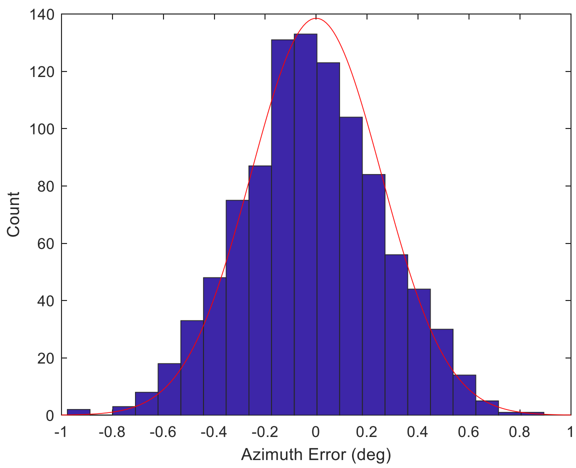


Fig. 11 distribution of predicted azimuth error and distribution of azimuth error from 1000 samples

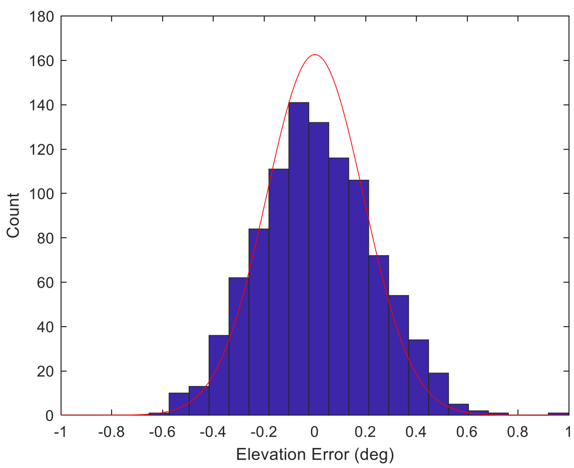


Fig. 12 distribution of predicted elevation error and distribution of elevation error from 1000 samples

Fig. 10 presents the uncertainty ellipse of vector QP computed from one pair of points P and Q (green line) and that computed from 1000 pairs of P and Q samples (red dashed line). Fig. 11 presents the distribution of predicted azimuth error (red solid line) and distribution of azimuth error from 1000 pairs of P&Q samples (histogram). Fig. 12 presents the distribution of predicted elevation error (red solid line) and distribution of elevation error from 1000 pairs of P&Q samples (histogram). In Fig 10–12, the standard deviation of both theodolite measurements is 0.1 degree for both azimuth and elevation measurements.

From Fig. 8–9, it is observed that, in the point determination, the ellipses with largest area inside the quadrangle of 1-SD perimeters of two theodolite measurements closely match the uncertainty ellipses calculated from Monte Carlo simulations. From Fig. 10, it is observed that, in the vector determination, the ellipses defined by the combined covariance matrices closely match the uncertainty ellipses calculated from Monte Carlo simulations. From Fig. 11, the developed model show a good prediction for azimuth error distribution that agrees with the distribution from 1000 samples of P&Q pairs for a test case. From Fig. 12, the developed model show a poor prediction for elevation error distribution when it is compared with the distribution from 1000 samples of P&Q pairs for a test case.

Fig. 13–14 present the analysis of azimuth determination using theodolite intersection method for the cases in which the object with different angles or different sizes at different distances (from the middle point between two theodolite). The results show that the uncertainty in azimuth prediction is smaller when the object is closer to the theodolite set-up. Note that the standard deviation of both theodolite measurements is 0.0014 degree (5 second of arc) for the results in Fig. 13–14.

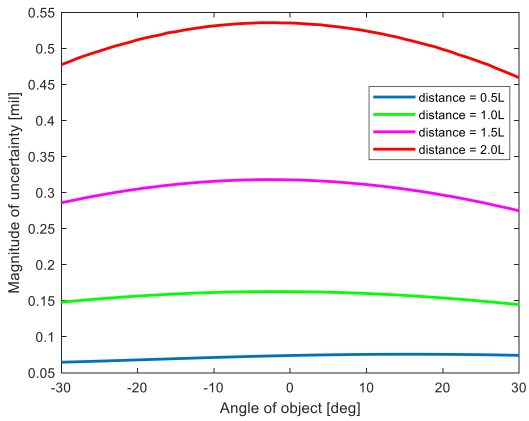


Fig. 13 prediction of azimuth uncertainty for object with different angles

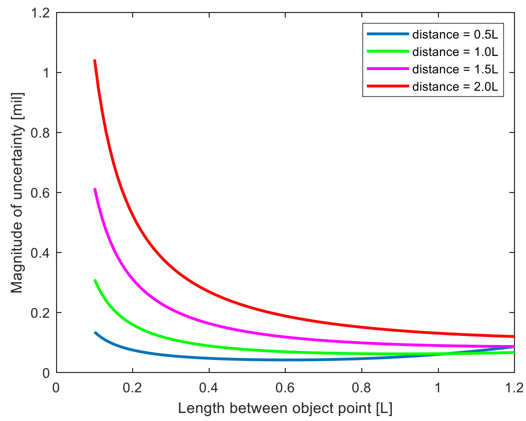


Fig. 14 prediction of azimuth uncertainty for object with different sizes

5 Conclusion

This paper presents the details of development of mathematical model which relies on geometric interpretation of covariance-matrix and simplification which separate horizontal and vertical component computations. The results show good prediction performance of the model for azimuth error distribution but show poor prediction performance for elevation error distribution. It is expected that the poor prediction performance in elevation determination comes from the simplification (separation) of horizontal and vertical computations. The ellipses with largest area inside the quadrangle of 1-SD measurements closely agree with those calculated from Monte Carlo simulations. Future work will focus on a method that combines uncertainty in horizontal and vertical determination and represents using an oval (ellipsoid) that fit inside 1-SD Hexahedron in three-dimension.

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7 References

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